Chen series and Atiyah-Singer theorem

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Abstract

The purpose of this work is to give a new and short proof of the Atiyah-Singer local index theorem for the Dirac operator on the spin bundle. This proof is obtained by using heat semigroups approximations based on the truncation of Brownian Chen series.

Contents

1 Introduction

The goal of this paper is to give a new and short proof of the local Atiyah-Singer index theorem by using approximations of heat semigroups. The heat equation approach to index theorems is not new: It was suggested by Atiyah-Bott [1] and McKean-Singer [18], and first carried out by Patodi [21] and Gilkey [13]. Bismut in [8] introduces stochastic methods based on Feynman-Kac formula. For probabilistic approaches that are mainly based on Bismut's ideas, we also refer to [15] and Chapter 7 of [14]. For a complete survey on (non probabilistic) heat equation methods for index theorems, we refer to the book [7].

However, in our approach we will see that the A-genus appears in a natural way, from purely local computations on approximations of heat semigroups. Our method relies on explicit approximations of the holonomy over the heat equation on vector bundles and unlike the other probabilistic approaches does not involve the Feynman-Kac formula nor the technics of stochastic differential geometry.

The idea is the following. Let $\mathbf{P}_t$ denote a heat semigroup. In recent works, see Baudoin [4] and Lyons-Victoir [17], by using Brownian Chen series, it has been pointed out that $\mathbf{P}_t$ admits a formal

representation as the expectation of the exponential of a random Lie series. The truncature of this Lie series leads to explicit approximations of $\mathbf{P}_t$. More precisely, one gets a family of operators $\mathbf{P}_t^N$, $N \geq 1$, such that (in the sup norm)

$$\mathbf{P}_t = \mathbf{P}_t^N + O(t^{\frac{N+1}{2}}), \quad t \to 0. \tag{1.1}$$

This point of view has been used in Lyons-Victoir [17] to provide cubature formulae on Wiener space that give efficient numerical approximations of solutions of heat equations.

Assume now that $\mathbf{P}_t$ is the heat semigroup associated with the Dirac operator on the Clifford module over a compact d-dimensional spin manifold, d even. From (1.1), we will classically deduce

$$\mathbf{Str}\ \mathbf{P}_t = \mathbf{Str}\ \mathbf{P}_t^d + O(t^{\frac{1}{2}}), \quad t \to 0,$$

where **Str** denotes the supertrace. The Lie structure that explicitly appears in $\mathbf{P}_t^d$ now leads to algebraic cancellations that imply

$$\mathbf{Str}\ \mathbf{P}_t^d = \mathbf{Str}\ \mathbf{P}_t^2 + O(t^{\frac{1}{2}}), \quad t \to 0.$$

Since, from McKean-Singer theorem, the supertrace of $\mathbf{P}_t$ has to be constant and equal to the index of the Dirac operator, the local index theorem follows from the easy computation of $\mathbf{Str}\ \mathbf{P}_t^2$.

The paper is organized as follows. In the first part, we survey results on random Chen series that are needed for the construction of approximations of heat semigroups. In the second part, we use these series to construct explicit approximations of the holonomy above general heat equations on vector bundles. Finally, in the third part, we develop the idea hinted above to provide a new short proof of the Atiyah-Singer local index theorem for the Dirac operator on the spin bundle.

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2 Chen series

We introduce here notations that will be used throughout the paper and survey some results on Chen series that will be later needed. Basic background on Chen series with respect to regular paths can be found in [10] and background on Chen series with respect to Brownian paths can be found in Chapter 1 of [4] (see also [11] and [17]). Let us note that the Chen series with respect to Brownian paths is also called, in the rough paths theory of Lyons, the signature of the Brownian motion.
Let $\mathbb{R}[[X_0, ..., X_d]]$ be the non commutative algebra over $\mathbb{R}$ of the formal series with $d+1$ indeterminates, that is the set of series

$$Y = \sum_{k \geq 0} \sum_{i_1,...,i_k} a_{i_1,...,i_k} X_{i_1} ... X_{i_k}.$$

The exponential of $Y \in \mathbb{R}[[X_0, ..., X_d]]$ is defined by

$$\exp(Y) = \sum_{k=0}^{+\infty} \frac{Y^k}{k!}.$$

We define the bracket between two elements U and V of $\mathbb{R}[[X_0, ..., X_d]]$ by

$$[U, V] = UV - VU.$$

If $I = (i_1, ..., i_k) \in \{0, ..., d\}^k$ is a word, we denote by X_I the commutator defined by

$$X_I = [X_{i_1}, [X_{i_2}, ..., [X_{i_{k-1}}, X_{i_k}]...].$$

We will denote by $\mathcal{S}_k$ the set of the permutations of $\{0, ..., k\}$. If $\sigma \in \mathcal{S}_k$, we denote $e(\sigma)$ the cardinality of the set

$$\{j \in \{0, ..., k-1\}, \sigma(j) > \sigma(j+1)\},$$

and $\sigma(I)$ the word $(i_{\sigma(1)}, ..., i_{\sigma(k)})$.
Let us now consider a d-dimensional standard Brownian motion $(B_t)_{t\geq 0} = (B_t^1, ..., B^d)_{t\geq 0}$. We use the convention that $B_t^0 = t$. If $I = (i_1, ...i_k) \in \{0, ..., d\}^k$ is a word with length k, the iterated Stratonovich integral

$$\int_{\Delta^k[0,t]} \circ dB^I = \int_{0\leq t_1 \leq ... \leq t_k \leq t} \circ dB_{t_1}^{i_1} \circ ... \circ dB_{t_k}^{i_k},$$

can be defined as the limit in p-variation, $p > 2$,

$$\lim_{n\to+\infty} \int_{\Delta^k[0,t]} (dB^n)^I,$$

where B^n denotes the piecewise linear interpolation of the paths of $(B_u)_{0\leq u\leq t}$ along the dyadic subdivision of $[0, t]$.
With the notation,

$$\Lambda_I(B)_t = \sum_{\sigma\in\mathcal{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \begin{pmatrix} k-1 \\ e(\sigma) \end{pmatrix}} \int_{\Delta^k[0,t]} \circ dB^{\sigma^{-1}(I)},$$

we have the following theorem:

Theorem 2.1

$$1 + \sum_{k=1}^{+\infty} \sum_{I\in\{0,1,...,d\}^k} \left(\int_{\Delta^k[0,t]} \circ dB^I \right) X_{i_1}...X_{i_k} = \exp\left(\sum_{k\geq 1} \sum_{I\in\{0,1,...,d\}^k} \Lambda_I(B)_t X_I \right), \; t \geq 0.$$

This theorem is due to Chen [10] and Strichartz [22] that prove that the above result holds for absolutely continuous paths. The result for Brownian paths is pointed out in Fliess [11]. And finally, Lyons [16], with rough paths theory, shows that it actually can be extended to very general paths.
If

$$Y = \sum_{k\geq 0} \sum_{I=(i_1,...,i_k)} a_{i_1,...,i_k} X_{i_1}...X_{i_k}$$

is a random series, that is if the coefficients are real random variables defined on a probability space, we will denote

$$\mathbb{E}(Y) = \sum_{k\geq 0} \sum_{I=(i_1,...,i_k)} \mathbb{E}(a_{i_1,...,i_k}) X_{i_1}...X_{i_k}$$

as soon as this expression makes sense, that is as soon as for every $I = (i_1, ..., i_k)$,

$$\mathbb{E}(\mid a_{i_1,...,i_k} \mid) < +\infty,$$

where $\mathbb{E}$ stands for the expectation.

Theorem 2.2 *(See [4], [17]). We have*

$$\exp\left(t\left(X_0+\frac{1}{2}\sum_{i=1}^{d}X_i^2\right)\right)=\mathbb{E}\left(\exp\left(\sum_{k\geq 1}\sum_{I\in\{0,1,\dots,d\}^k}\Lambda_I(B)_t X_I\right)\right),\quad t\geq 0.$$

3 Approximation of elliptic heat kernels on vector bundles

In the spirit of Azencott [3], Ben Arous [6], Castell [9] and Lyons-Victoir [17], we use in this section Brownian Chen series in order to provide efficient approximations of heat semigroups and corresponding heat kernels. The idea is to truncate the Lie series that appear in the formal representation of the heat semigroup given by Theorem 2.2. As we shall see, this truncation that has already found applications for cubature formulae [17], is also particularly efficient to approximate the holonomy over the heat equation in a vector bundle.

Let $\mathbb{M}$ be a d-dimensional compact smooth Riemannian manifold and let $\mathcal{E}$ be a finite-dimensional vector bundle over $\mathbb{M}$. We denote by $\Gamma(\mathbb{M},\mathcal{E})$ the space of smooth sections. Let now ∇ denote a connection on $\mathcal{E}$.
We consider the following linear partial differential equation

$$\frac{\partial\Phi}{\partial t}=\mathcal{L}\Phi,\quad \Phi(0,x)=f(x), \tag{3.2}$$

where $\mathcal{L}$ is an operator on $\mathcal{E}$ that can be written

$$\mathcal{L}=\nabla_0+\frac{1}{2}\sum_{i=1}^{d}\nabla_i^2,$$

with

$$\nabla_i=\mathcal{F}_i+\nabla_{V_i},\quad 0\leq i\leq d,$$

the V_i's being smooth vector fields on $\mathbb{M}$ and the $\mathcal{F}_i$'s being smooth potentials (that is sections of the bundle $\mathbf{End}(\mathcal{E})$). It is known that the solution of (3.2) can be written

$$\Phi(t,x)=(e^{t\mathcal{L}}f)(x)=\mathbf{P}_t f(x).$$

If $I\in\{0,1,...,d\}^k$ is a word, we denote

$$\nabla_I=[\nabla_{i_1},[\nabla_{i_2},...,[\nabla_{i_{k-1}},\nabla_{i_k}]...].$$

and

$$d(I)=k+n(I),$$

where $n(I)$ is the number of 0 in the word I.
For $N\geq 1$, let us consider

$$\mathbf{P}_t^N=\mathbb{E}\left(\exp\left(\sum_{I,d(I)\leq N}\Lambda_I(B)_t\nabla_I\right)\right).$$

For instance

$$\mathbf{P}_t^1 = \mathbb{E}\left(\exp\left(\sum_{i=1}^d B_t^i \nabla_i\right)\right),$$

and

$$\mathbf{P}_t^2 = \mathbb{E}\left(\exp\left(\sum_{i=0}^d B_t^i \nabla_i + \frac{1}{2}\sum_{1\le i<j\le d}\int_0^t B_s^i dB_s^j - B_s^j dB_s^i [\nabla_i, \nabla_j]\right)\right).$$

The meaning of this last notation is the following. If $f \in \Gamma(\mathbb{M}, \mathcal{E})$, then $(\mathbf{P}_t^N f)(x) = \mathbb{E}(\Psi(1, x))$, where $\Psi(\tau, x)$ is the solution of the first order partial differential equation with random coefficients:

$$\frac{\partial \Psi}{\partial \tau}(\tau, x) = \sum_{I, d(I)\le N} \Lambda_I(B)_t (\nabla_I \Psi)(\tau, x), \quad \Psi(0, x) = f(x).$$

Let us consider the following family of norms: If $f \in \Gamma(\mathbb{M}, \mathcal{E})$, for $k \ge 0$,

$$\parallel f \parallel_k = \sup_{0\le l\le k} \sup_{0\le i_1,\cdots,i_l\le d} \sup_{x\in\mathbb{M}} \parallel \nabla_{i_1}\cdots\nabla_{i_l} f(x) \parallel .$$

Theorem 3.1 *Let $N \ge 1$ and $k \ge 0$. For $f \in \Gamma(\mathbb{M}, \mathcal{E})$,*

$$\parallel \mathbf{P}_t f - \mathbf{P}_t^N f \parallel_k = O\left(t^{\frac{N+1}{2}}\right), \quad t \to 0.$$

Proof.
First, by using the scaling property of Brownian motion and expanding out the exponential with Taylor formula we obtain

$$\exp\left(\sum_{I, d(I)\le N} \Lambda_I(B)_t \nabla_I\right) f = \left(\sum_{k=0}^N \frac{1}{k!}\left(\sum_{I, d(I)\le N} \Lambda_I(B)_t \nabla_I\right)^k\right) f + t^{\frac{N+1}{2}} \mathbf{R}_N^1(t),$$

where the remainder term $\mathbf{R}_N^1(t)$ is such that $\mathbb{E}\left(\parallel \mathbf{R}_N^1(t) \parallel_k\right)$ is bounded when $t \to 0$. We now observe that, due to Theorem 2.1, the rearrangement of terms in the previous formula gives

$$\left(\sum_{k=0}^N \frac{1}{k!}\left(\sum_{I, d(I)\le N} \Lambda_I(B)_t \nabla_I\right)^k\right) f = f + \sum_{I, d(I)\le N} \int_{\Delta^{|I|}[0,t]} \circ dB^I \nabla_{i_1} ... \nabla_{i_{|I|}} f + t^{\frac{N+1}{2}} \mathbf{R}_N^2(t),$$

where $\mathbb{E}\left(\parallel \mathbf{R}_N^2(t) \parallel_k\right)$ is bounded when $t \to 0$. Therefore

$$\exp\left(\sum_{I, d(I)\le N} \Lambda_I(B)_t \nabla_I\right) f = f + \sum_{I, d(I)\le N} \int_{\Delta^{|I|}[0,t]} \circ dB^I \nabla_{i_1} ... \nabla_{i_{|I|}} f + t^{\frac{N+1}{2}} \mathbf{R}_N^3(t),$$

and

$$\mathbf{P}_t^N f = f + \sum_{I, d(I)\le N} \mathbb{E}\left(\int_{\Delta^{|I|}[0,t]} \circ dB^I\right) \nabla_{i_1} ... \nabla_{i_{|I|}} f + t^{\frac{N+1}{2}} \mathbb{E}\left(\mathbf{R}_N^3(t)\right),$$

where $\mathbb{E}\left(\| \mathbf{R}_N^3(t) \|_k\right)$ is bounded when $t \to 0$. We now have to compute the expectation of iterated Stratonovitch integrals. An easy computation shows that if $\mathcal{I}_n$ is the set of words with length n obtained by all the possible concatenations of the words

$$\{0\}, \{(i,i)\}, \quad i \in \{1, ..., d\},$$

1. If $I \notin \mathcal{I}_n$ then

$$\mathbb{E}\left(\int_{\Delta^n[0,t]} \circ dB^I\right) = 0;$$

2. If $I \in \mathcal{I}_n$ then

$$\mathbb{E}\left(\int_{\Delta^n[0,t]} \circ dB^I\right) = \frac{t^{\frac{n+n(I)}{2}}}{2^{\frac{n-n(I)}{2}}\left(\frac{n+n(I)}{2}\right)!},$$

   where $n(I)$ is the number of 0 in I (observe that since $I \in \mathcal{I}_n$, n and $n(I)$ necessarily have the same parity).

We conclude therefore

$$\| \mathbf{P}_t^N f - \sum_{k \leq \frac{N+1}{2}} \frac{t^k}{k!} \mathcal{L}^k f \|_k = O\left(t^{\frac{N+1}{2}}\right).$$

Since it is known that

$$\| \mathbf{P}_t f - \sum_{k \leq \frac{N+1}{2}} \frac{t^k}{k!} \mathcal{L}^k f \|_k = O\left(t^{\frac{N+1}{2}}\right),$$

the theorem is proved.

□

Let us, here, assume that the operator $\mathcal{L}$ is elliptic at $x_0 \in \mathbb{M}$ in the sense that $(V_1(x_0), ..., V_d(x_0))$ is an orthonormal basis of the tangent space at x_0. In that case, $\mathbf{P}_t$ is known to admit a smooth Schwartz kernel at x_0. That is, there exists a smooth map

$$p(x_0, \cdot) : \mathbb{R}_{>0} \to \Gamma(\mathbb{M}, \mathbf{Hom}(\mathcal{E}))$$

Theorem 3.2 *Let $N \geq 1$. There exists a map*

$$p^N(x_0, \cdot) : \mathbb{R}_{>0} \to \Gamma(\mathbb{M}, \mathbf{Hom}(\mathcal{E}))$$

such that for $f \in \Gamma(\mathbb{M}, \mathcal{E})$,

$$(\mathbf{P}_t^N f)(x_0) = \int_{\mathbb{M}} p_t^N(x_0, y) f(y) dy.$$

Moreover,

$$p_t(x_0, x_0) = p_t^N(x_0, x_0) + 0\left(t^{\frac{N+1-d}{2}}\right).$$

Proof.
The proof is not simple. We shall proceed in several steps. In a first step, we shall show the existence of a kernel at x_0 for $\mathbf{P}_t^N$ acting on functions. In a second step we shall deduce by parallel transport, the existence of $p^N(x_0,\cdot)$. And finally, we shall prove the required estimate.

First step:
Let us define,

$$\mathbf{Q}_t^N = \mathbb{E}\left(\exp\left(\sum_{I,d(I)\le N} \Lambda_I(B)_t V_I\right)\right).$$

In order to show that $\mathbf{Q}_t^N$ admits a kernel at x_0, we show that the stochastic process

$$Z_t^N = \exp\left(\sum_{I,d(I)\le N} \Lambda_I(B)_t V_I\right)(x_0)$$

has a density with respect to the Riemannian measure of $\mathbb{M}$. To this end, from
Though the family $(\mathbf{Q}_t^N)_{t\ge 0}$ is not a semigroup in general, the idea is to factorize $(\mathbf{Q}_t^N)_{t\ge 0}$ via a semigroup acting on functions and defined on a N-step nilpotent group.

Up to isomorphism, there exists a unique simply connected and nilpotent Lie group $\mathbb{G}_N$ with Lie algebra $\mathfrak{g}_N$ such that $\mathfrak{g}_N$ is isomorphic by a Lie algebra morphism Φ to the Lie algebra $\mathbb{R}[[X_0,...,X_d]]$ quotiented by the relations

$$\{X_I = 0, d(I) \ge N+1\}.$$

We can write a stratification

$$\mathfrak{g}_N = \mathcal{V}_1 \oplus ... \oplus \mathcal{V}_N,$$

where

$$\mathcal{V}_k = \mathbf{span}\{U_I, d(I) = k\}, \quad U_i = \Phi(X_i).$$

The canonical sublaplacian on $\mathbb{G}_N$ is

$$\Delta_N = U_0 + \frac{1}{2}\sum_{i=1}^d U_i^2.$$

and, let us observe that the bracket generating condition is satisfied at each point, so that due to Hörmander's theorem, Δ_N is hypoelliptic.
According to the Chow's theorem in Carnot groups (see for instance Theorem 2.4 in [4]), for $x_0 \in \mathbb{M}$, there exists a unique smooth map

$$\pi_{x_0}^N : \mathbb{G}_N \to \mathbb{M}$$

such that for any piecewise smooth path $x : [0,1] \to \mathbb{R}^d$ and any smooth function $f : \mathbb{M} \to \mathbb{R}$,

$$\left[\exp\left(\sum_{I,d(I)\le N} \Lambda_I(x)_1 U_I\right)(f\circ \pi_{x_0}^N)\right](1_{\mathbb{G}_N}) = \left[\exp\left(\sum_{I,d(I)\le N} \Lambda_I(x)_1 V_I\right)f\right](x_0).$$

Since

$$Y_t = \exp\left(\sum_{I,d(I)\le N} \Lambda_I(B)_t U_I\right)(1_{\mathbb{G}_N})$$

is seen to be the solution of the following stochastic differential equation defined on $\mathbb{G}_N$

$$Y_t = 1_{\mathbb{G}_N} + \int_0^t Y_s\left(U_0 ds + \sum_{i=1}^d U_i \circ dB_s^i\right).$$

we get the following factorization of $\mathbf{Q}_t^N$ in $\mathbb{G}_N$: For every $x_0 \in \mathbb{M}$ and every smooth $f : \mathbb{M} \to \mathbb{R}$,

$$(\mathbf{Q}_t^N f)(x_0) = e^{t\Delta_N}(f \circ \pi_{x_0}^N)(1_{\mathbb{G}_N}), \quad t \ge 0.$$

Since Δ_N is subelliptic, $e^{t\Delta_N}$ has a smooth Schwartz kernel. But now, since $\mathcal{L}$ is elliptic at x_0, the differential of $\pi_{x_0}^N$ has maximal rank at $1_{\mathbb{G}_N}$. We get therefore the existence of $\mathcal{O}_{x_0}$ and of a smooth $q^N(x_0,\cdot) : \mathbb{R}_{>0} \times \mathcal{O}_{x_0} \to \mathbb{R}_{\ge 0}$ such that for every every smooth $f : \mathbb{M} \to \mathbb{R}$ with compact support included in $\mathcal{O}_{x_0}$,

$$(\mathbf{Q}_t^N f)(x_0) = \int_{\mathbb{M}} q_t^N(x_0, y) f(y) dy.$$

Second step:
For $t > 0$, let us consider the operator $\Theta_t^N(x_0)$ defined on $\Gamma(\mathbb{M}, \mathcal{E})$ by the property that for $\eta \in \Gamma(\mathbb{M}, \mathcal{E})$ and $y \in \mathcal{O}_{x_0}$,

$$(\Theta_t^N(x_0)\eta)(y) = \mathbb{E}\left(\left[\exp\left(\sum_{I,d(I)\le N} \Lambda_I(B)_t \nabla_I\right)\eta\right](x_0) \middle| \exp\left(\sum_{I,d(I)\le N} \Lambda_I(B)_t V_I\right)(x_0) = y\right).$$

We claim that $\Theta_t^N(x_0)$ is actually a potential, that is a smooth section of the bundle $\mathbf{End}(\mathcal{E})$. For that, we have to show that for every smooth $f : \mathbb{M} \to \mathbb{R}$ and every $\eta \in \Gamma(\mathbb{M}, \mathcal{E})$, $y \in \mathcal{O}_{x_0}$,

$$(\Theta_t(x_0) f\eta)(y) = f(y)(\Theta_t^N(x_0)\eta)(y).$$

If f is a smooth function on $\mathbb{M}$, we denote by $\mathcal{M}_f$ the operator on $\Gamma(\mathbb{M}, \mathcal{E})$ that acts by multiplication by f. Due to the Leibniz rule for connections, we have for any word I:

$$[\nabla_I, \mathcal{M}_f] = \mathcal{M}_{V_I f}.$$

Consequently,

$$\left[\sum_{I,d(I)\le N} \Lambda_I(B)_t \nabla_I, \mathcal{M}_f\right] = \mathcal{M}_{\sum_{I,d(I)\le N} \Lambda_I(B)_t V_I}.$$

The above commutation property implies the following one:

$$\exp\left(\sum_{I,d(I)\le N} \Lambda_I(B)_t \nabla_I\right)\mathcal{M}_f = \mathcal{M}_{\exp\left(\sum_{I,d(I)\le N} \Lambda_I(B)_t V_I\right) f} \exp\left(\sum_{I,d(I)\le N} \Lambda_I(B)_t \nabla_I\right).$$

Therefore,

$$[\Theta_t^N(x_0), \mathcal{M}_f] = 0,$$

so that $\Theta_t^N(x_0) \in \Gamma(\mathbb{M}, \mathbf{End}(\mathcal{E}))$. We can now conclude with the disintegration formula that for every $\eta \in \Gamma(\mathbb{M}, \mathcal{E})$ with compact support included in $\mathcal{O}_{x_0}$,

$$(\mathbf{P}_t^N \eta)(x_0) = \int_{\mathbb{M}} p_t^N(x_0, y)\eta(y)dy,$$

with

$$p_t^N(x_0, \cdot) = q_t^N(x_0, \cdot)\Theta_t^N(x_0).$$

Final step:
Let us now turn to the proof of the pointwise estimate

$$p_t(x_0, x_0) = p_t^N(x_0, x_0) + O\left(t^{\frac{N+1-d}{2}}\right), \quad t \to 0.$$

Let $y \in \mathbb{M}$ sufficiently close to x_0. Since $\mathcal{L}$ is elliptic at x_0, it is known that $p_t(x_0, y)$ admits a development

$$p_t(x_0, y) = \frac{e^{-\frac{d^2(x_0,y)}{2t}}}{(2\pi t)^{d/2}} \left(\sum_{k=0}^{N} \Psi_k(x_0, y)t^k + t^{\frac{N+1}{2}} \mathbf{R}_N(t, x_0, y) \right), \tag{3.3}$$

where the remainder term $\mathbf{R}_N(t, x_0, y)$ is bounded when $t \to 0$, $\Psi_k(x_0, \cdot)$ is a section of $\mathbf{End}(\mathcal{E})$ defined around x_0 and $d(\cdot, \cdot)$ is the distance defined around x_0 by the vector fields $V_1, ..., V_d$. By using the fact that for every smooth $f : \mathbb{M} \to \mathbb{R}$,

$$(\mathbf{Q}_t^N f)(x_0) = e^{t\Delta_N}(f \circ \pi_{x_0}^N)(1_{\mathbb{G}_N}), \quad t \geq 0,$$

and classical results for asymptotic development in small times of subelliptic heat kernels (see for instance [5]), we get for $q_t^N(x_0, y)$ a development that is similar to (3.3). For $\Theta_t^N(x_0)$, the scaling property of Brownian motion implies that we have a short-time asymptotics in powers $t^{\frac{k}{2}}$, $k \in \mathbb{N}$. Since,

$$p_t^N(x_0, \cdot) = q_t^N(x_0, \cdot)\Theta_t^N(x_0),$$

we deduce that

$$p_t^N(x_0, y) = \frac{e^{-\frac{d^2(x_0,y)}{2t}}}{(2\pi t)^{d/2}} \left(\sum_{k=0}^{N} \tilde{\Psi}_k(x_0, y)t^k + t^{\frac{N+1}{2}} \tilde{\mathbf{R}}_N(t, x_0, y) \right),$$

where the remainder term $\tilde{\mathbf{R}}_N(t, x_0, y)$ is bounded when $t \to 0$. With Theorem 3.1, we obtain that $\Psi_k = \tilde{\Psi}_k$, $k = 0, ..., N$, and the required estimate easily follows. $\square$

If $I \in \{0, 1, ..., d\}^k$ is a word, we denote

$$\mathcal{F}_I = \nabla_I - \nabla_{V_I} \in \Gamma(\mathbb{M}, \mathbf{End}(\mathcal{E})).$$

Corollary 3.3 *For $N \geq 1$, when $t \to 0$,*

$$p_t(x_0, x_0) = q_t^N(x_0)\mathbb{E}\left(\exp\left(\sum_{I,d(I)\leq N} \Lambda_I(B)_t \mathcal{F}_I(x_0)\right) \,\middle|\, \sum_{I,d(I)\leq N} \Lambda_I(B)_t V_I(x_0) = 0\right) + O\left(t^{\frac{N+1-d}{2}}\right),$$

where $q_t^N(x_0)$ is the density at 0 of the random variable $\sum_{I,d(I)\leq N} \Lambda_I(B)_t V_I(x_0)$.

4 The local index theorem for the Dirac operator on the spin bundle

The Atiyah-Singer index theorem for the Dirac operator on the spin bundle as proved in [2], is the following:

Theorem 4.1 *Let $\mathbb{M}$ be a compact, d-dimensional spin manifold, with d even. Let* $\mathbf{D}$ *be the Dirac operator on the spin bundle of* $\mathbb{M}$*. Then*

$$\mathbf{ind}(\mathbf{D}) = \left(\frac{1}{2i\pi}\right)^{\frac{d}{2}} \int_{\mathbb{M}} [A(\mathbb{M})]_d,$$

where $[A(\mathbb{M})]_d$ is the volume form on $\mathbb{M}$ obtained by taking the d-form piece of the A-genus

$$A(\mathbb{M}) = \det\left(\frac{\Omega}{2\sinh\frac{1}{2}\Omega}\right)^{\frac{1}{2}},$$

and Ω is the Riemannian curvature form defined in local orthonormal frame e_i with dual frame e_i^* by

$$\Omega = \frac{1}{2}\sum_{1\le i,j\le d} R(e_i,e_j)e_i^* \wedge e_j^*,$$

with R, Riemannian curvature.

Before we turn to the proof. Let us first recall some linear algebra constructions as can be found in Chapter 3 of [7].

Let V be an oriented d dimensional Euclidean space. We assume that the dimension d is even. The Clifford algebra $\mathbf{Cl}(V)$ over V is the algebra

$$\mathbf{T}(V) = \mathbb{R} \oplus V \oplus (V\otimes V) \oplus \cdots$$

quotiented by the relations

$$u\otimes v + v\otimes u + 2\langle u,v\rangle 1 = 0. \tag{4.4}$$

Let $e_1, ..., e_d$ be an oriented basis of V. The family

$$e_{i_1}...e_{i_k}, \quad 0\le k\le d, \quad 1\le i_1 < ... < i_k \le d,$$

forms a basis of $\mathbf{Cl}(V)$ that is therefore of dimension 2^d. In $\mathbf{T}(V)$ we can distinguish elements that are even from elements that are odd. This leads to a decomposition:

$$\mathbf{Cl}(V) = \mathbf{Cl}^-(V) \oplus \mathbf{Cl}^+(V),$$

with $V\subset \mathbf{Cl}^-(V)$.
A Clifford module is a vector space E over $\mathbb{R}$ (or $\mathbb{C}$) that is also a $\mathbf{Cl}(V)$-module and that admits a direct sum decomposition

$$E = E^- \oplus E^+$$

with

$$\mathbf{Cl}^-(V)\cdot E^- \subset E^-, \quad \mathbf{Cl}^+(V)\cdot E^+ \subset E^+.$$

It can be shown that there is a unique Clifford module S, called the spinor module over V such that:

$$\mathbf{End}(S) \simeq \mathbb{C}\otimes \mathbf{Cl}(V).$$

In particular $\dim S = 2^{\frac{d}{2}}$. There is therefore a natural notion of supertrace on $\mathbf{Cl}(V)$ that is given by

$$\mathbf{Str}\ a = \mathbf{Tr}_{S^+}\ a - \mathbf{Tr}_{S^-}\ a,$$

where $a \in \mathbf{Cl}(V)$ is seen as an element of $\mathbf{End}(S)$. If

$$a = \sum_{k=0}^{d} \sum_{1\le i_1<\ldots<i_k\le d} a_{i_1\cdots i_k} e_{i_1}\ldots e_{i_k},$$

then we have

$$\mathbf{Str}\ a = \left(\frac{2}{i}\right)^{\frac{d}{2}} a_{1\cdots d}. \tag{4.5}$$

If $\psi \in \mathfrak{so}(V)$, that is if $\psi : V \to V$ is a skew-symmetric map, we define

$$D\psi = \frac{1}{2}\sum_{1\le i<j\le d} \langle \psi(e_i), e_j\rangle e_i e_j \in \mathbf{Cl}(V),$$

and observe that $D[\psi_1,\psi_2] = [D\psi_1, D\psi_2]$. The set $\mathbf{Cl}^2(V) = D\mathfrak{so}(V)$ is therefore a Lie algebra. The Lie group $\mathbf{Spin}(V)$ is the group obtained by exponentiating $\mathbf{Cl}^2(V)$ inside the Clifford algebra $\mathbf{Cl}(V)$; It is the two-fold universal covering of the orthogonal group $\mathbf{SO}(V)$. It can also be described as the set of $a \in \mathbf{Cl}(V)$ such that:

$$a = v_1\ldots v_{2k}, \quad 1\le k \le \frac{d}{2}, \quad v_i \in V, \quad \parallel v_i \parallel = 1.$$

We now come back to differential geometry and carry the above construction on the cotangent spaces of a spin manifold.

So, let $\mathbb{M}$ be a compact d-dimensional, Riemannian and oriented manifold. We assume that d is even. We furthermore assume that $\mathbb{M}$ admits a spin structure: That is, there exists a principal bundle on $\mathbb{M}$ with structure group $\mathbf{Spin}(\mathbb{R}^d)$ such that the bundle charts are compatible with the universal covering $\mathbf{Spin}(\mathbb{R}^d) \to \mathbf{SO}(\mathbb{R}^d)$. This bundle will be denoted $\mathcal{SP}(\mathbb{M})$ and π will denote the canonical surjection. The spin bundle $\mathcal{S}$ over $\mathbb{M}$ is the vector bundle such that for every $x\in\mathbb{M}$, $\mathcal{S}_x$ is the spinor module over the cotangent space $\mathbf{T}^*_x\mathbb{M}$. At each point x, there is therefore a natural action of $\mathbf{Cl}(\mathbf{T}^*_x\mathbb{M}) \simeq \mathbf{End}(\mathcal{S}_x)$; this action will be denoted by $\mathbf{c}$.

On $\mathcal{S}$, there is a canonical elliptic first-order differential operator called the Dirac operator and denoted $\mathbf{D}$. In a local orthonormal frame e_i, with dual frame e_i^*, we have

$$\mathbf{D} = \sum_i c(e_i^*)\nabla_{e_i},$$

where ∇ is the Levi-Civita connection. We have an analogue of Weitzenböck formula which is the celebrated Lichnerowicz formula (see Theorem 3.52 in [7]):

$$\mathbf{D}^2 = \Delta + \frac{s}{4},$$

where s is the scalar curvature of $\mathbb{M}$ and Δ is given in a local orthonormal frame e_i by

$$\Delta = -\sum_{i=1}^{d} (\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i} e_i}).$$

After these preliminaries, we can now turn to the proof of the local index theorem for $\mathbf{D}$.

The first crucial step in all heat equations approaches of index theorems is the McKean-Singer type formula (see [18] and Theorem 3.50 in [7]):

$$\mathbf{ind}(\mathbf{D}) = \mathbf{Str}(\mathbf{P}_t) = \int_{\mathbb{M}} \mathbf{Str}\ p_t(x,x)dx, \quad t > 0,$$

where $\mathbf{P}_t = e^{-\frac{1}{2}t\mathbf{D}^2}$, p_t is the corresponding Schwartz kernel, and dx is the Riemannian volume form. By using the results of Section 3 we now show that we actually have

$$\lim_{t\to 0} \mathbf{Str}\ p_t(x,x)dx = \left(\frac{1}{2i\pi}\right)^{\frac{d}{2}} [A(\mathbb{M})]_d(x).$$

This last statement is first due to Patodi [21] and Gilkey [13] and implies the index theorem.

Let us fix $x_0 \in \mathbb{M}$ once time for all in what follows. Let e_i be a synchronous local orthonormal frame centered at x_0 with dual frame e_i^*. If needed, with a cut-off function, we extend smoothly the vector field e_i to be zero outside a neigborhood of x_0. At the center x_0 of the frame, we have:

$$\Delta = -\sum_{i=1}^{d} \nabla_{e_i}\nabla_{e_i}$$

and

$$[\nabla_{e_i}, \nabla_{e_j}] = R(e_i, e_j),$$

where R is the Riemannian curvature.
For $t > 0$, let $\Theta_t(x_0) \in \mathbf{End}(\mathcal{S}_{x_0}) \simeq \mathbf{Cl}(\mathbf{T}^*_{x_0}\mathbb{M})$ be the Clifford element such that for every smooth section η of $\mathcal{S}$,

$$\Theta_t(x_0)\eta(x_0) = \mathbb{E}\left(\left[\exp\left(\sum_{I, d(I)\le d} \Lambda_I(B)_t \nabla_I \right) \eta \right](x_0) \middle| \exp\left(\sum_{I, |I|\le d, 0\notin I} \Lambda_I(B)_t e_I \right)(x_0) = x_0 \right),$$

where in the above summation, we use the convention that ∇_0 is the multiplication operator by $-\frac{s}{8}$ and $\nabla_i = \nabla_{e_i}$, $1 \le i \le d$.

Proposition 4.2

$$\lim_{t\to 0} \frac{\mathbf{Str}\ \Theta_t(x_0)}{t^{d/2}} = \frac{1}{2^{d/2}(d/2)!} \mathbf{Str}\ \mathbb{E}\left(\left(\sum_{1\le i<j\le d} DR(e_i, e_j) \int_0^1 B_s^i dB_s^j - B_s^j dB_s^i \right)^{\frac{d}{2}} \middle| B_1 = 0 \right)$$

Proof.
First, let us observe that due to the scaling property of Brownian motion, for every smooth section η of $\mathcal{S}$,

$$\Theta_t(x_0)\eta(x_0) = \mathbb{E}\left(\left[\exp\left(\sum_{I,d(I)\leq d} t^{d(I)/2}\Lambda_I(B)_1\nabla_I\right)\eta\right](x_0) \middle| \exp\left(\sum_{I,|I|\leq d,0\notin I} t^{d(I)/2}\Lambda_I(B)_1 e_I\right)(x_0) = x_0\right)$$

Let us now rewrite more explicitly $\Theta_t(x_0)$ as a Clifford element.
If $1 \leq i < j \leq d$, we have at x_0

$$[\nabla_i, \nabla_j] = \mathbf{c}\left(DR(e_i, e_j)\right)$$

with

$$DR(e_i, e_j) = \frac{1}{2}\sum_{1\leq k<l\leq d} \langle R(e_i, e_j)e_k, e_l\rangle e_k^* e_l^* \in \mathbf{Cl}^2(\mathbf{T}^*_{x_0}\mathbb{M})$$

Since the Levi-Civita connection is a Clifford connection, if $1 \leq i < j < k \leq d$, we have at x_0,

$$\begin{aligned}
[\nabla_i, [\nabla_j, \nabla_k]] &= [\nabla_i, \nabla_{[e_j,e_k]} + \mathbf{c}\left(DR(e_j, e_k)\right)] \\
&= \nabla_{[e_i,[e_j,e_k]]} + \mathbf{c}\left(DR(e_i, [e_j, e_k]) + \nabla_i DR(e_j, e_k)\right).
\end{aligned}$$

and we observe that $DR(e_i, [e_j, e_k]) + \nabla_i DR(e_j, e_k) \in \mathbf{Cl}^2(\mathbf{T}^*_{x_0}\mathbb{M})$.
More generally, a recurrence procedure shows that if $1 \leq i_1 < ... < i_k \leq d$, then at x_0,

$$\nabla_I - \nabla_{e_I} = c(\mathcal{F}_I),$$

where $\mathcal{F}_I \in \mathbf{Cl}^2(\mathbf{T}^*_{x_0}\mathbb{M})$.
If $0 \in I$, then it is seen that, at x_0, ∇_I acts by multiplication with a scalar. Therefore

$$\Theta_t(x_0) = \mathbb{E}\left(\mathbf{c}\left(\exp\left(X_t + \sum_{I,|I|\leq d,0\notin I} t^{|I|/2}\Lambda_I(B)_1\mathcal{F}_I\right)\right) \middle| \exp\left(\sum_{I,|I|\leq d,0\notin I} t^{|I|/2}\Lambda_I(B)_1 e_I\right)(x_0) = x_0\right),$$

where X_t is scalar term such that $X_0 = 0$. We deduce

$$\mathbf{Str}\ \Theta_t(x_0) = \mathbb{E}\left(e^{X_t}\mathbf{Str}\ \exp\left(\sum_{I,|I|\leq d,0\notin I} t^{|I|/2}\Lambda_I(B)_1\mathcal{F}_I\right) \middle| \exp\left(\sum_{I,|I|\leq d,0\notin I} t^{|I|/2}\Lambda_I(B)_1 e_I\right)(x_0) = x_0\right)$$

and

$$\mathbf{Str}\ \Theta_t(x_0) \sim_{t\to 0} \mathbb{E}\left(\mathbf{Str}\ \exp\left(\sum_{I,|I|\leq d,0\notin I} t^{|I|/2}\Lambda_I(B)_1\mathcal{F}_I\right) \middle| B_1 = 0\right)$$

But now, since $\mathcal{F}_I \in \mathbf{Cl}^2(\mathbf{T}^*_{x_0}\mathbb{M})$, according to (4.5), for any $k < \frac{d}{2}$,

$$\mathbf{Str}\ \left(\sum_{I,|I|\leq d,0\notin I} t^{|I|/2}\Lambda_I(B)_1\mathcal{F}_I\right)^k = 0,$$

and, when $t \to 0$,

$$
\begin{aligned}
&\mathbb{E}\left(\mathbf{Str}\ \left(\sum_{I,|I|\le d, 0\notin I} t^{|I|/2}\Lambda_I(B)_1\mathcal{F}_I\right)^{\frac{d}{2}} \Bigg| B_1 = 0\right)\\
=&t^{d/2}\mathbb{E}\left(\mathbf{Str}\ \left(\frac{1}{2}\sum_{1\le i<j\le d} DR(e_i,e_j)\int_0^1 B_s^i dB_s^j - B_s^j dB_s^i\right)^{\frac{d}{2}} \Bigg| B_1 = 0\right) + O(t^{\frac{d+1}{2}}).
\end{aligned}
$$

We conclude therefore

$$
\lim_{t\to 0}\frac{\mathbf{Str}\ \Theta_t(x_0)}{t^{d/2}} = \mathbb{E}\left(\mathbf{Str}\ \frac{1}{(d/2)!}\left(\frac{1}{2}\sum_{1\le i<j\le d} DR(e_i,e_j)\int_0^1 B_s^i dB_s^j - B_s^j dB_s^i\right)^{\frac{d}{2}} \Bigg| B_1 = 0\right).
$$

□

We can now obtain the required limit:

Theorem 4.3

$$
\lim_{t\to 0}\mathbf{Str}\ p_t(x_0,x_0)dx_0 = \left(\frac{1}{2i\pi}\right)^{\frac{d}{2}}[A(\mathbb{M})]_d(x_0).
$$

Proof. By Theorem **??** and Proposition 4.2, we get

$$
\begin{aligned}
&\lim_{t\to 0}\mathbf{Str}\ p_t(x_0,x_0)dx_0\\
=&\frac{1}{(4\pi)^{d/2}(d/2)!}\mathbf{Str}\ \mathbb{E}\left(\left(\sum_{1\le i<j\le d} DR(e_i,e_j)\int_0^1 B_s^i dB_s^j - B_s^j dB_s^i\right)^{\frac{d}{2}} \Bigg| B_1 = 0\right) e_1^*\wedge ...\wedge e_d^*.
\end{aligned}
$$

From (4.5) and from the expression of $DR(e_i,e_j)$, the above expression is also equal to the d-form piece of

$$
\frac{1}{(2i\pi)^{d/2}(d/2)!}\mathbb{E}\left(\left(\sum_{1\le i<j\le d}\frac{1}{2}\Omega(e_i,e_j)\int_0^1 B_s^i dB_s^j - B_s^j dB_s^i\right)^{\frac{d}{2}} \Bigg| B_1 = 0\right).
$$

This last expression is also the d-form piece of

$$
\frac{1}{(2i\pi)^{d/2}}\mathbb{E}\left(\exp\left(\sum_{1\le i<j\le d}\frac{1}{2}\Omega(e_i,e_j)\int_0^1 B_s^i dB_s^j - B_s^j dB_s^i\right) \Bigg| B_1 = 0\right),
$$

which turns out to be the d-form piece of the A-genus

$$
\frac{1}{(2i\pi)^{d/2}}A(\mathbb{M}) = \frac{1}{(2i\pi)^{d/2}}\det\left(\frac{\Omega}{2\sinh\frac{1}{2}\Omega}\right)^{\frac{1}{2}},
$$

from Lévy's area formula (see for instance Lemma 7.6.6 in [14]). □

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